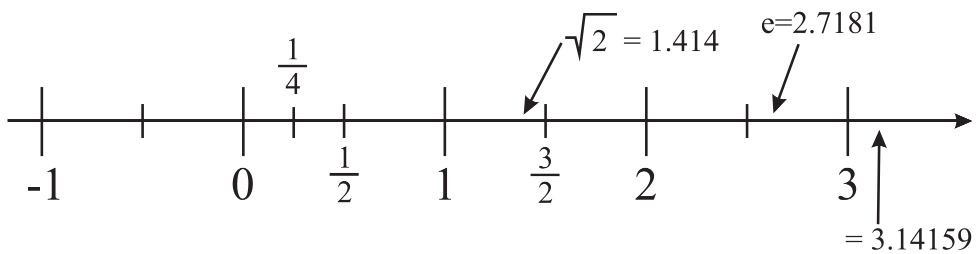
2.1 Complex numbers: Definition

(i) integers: $\{\ldots,-3,-2,-1,\underbrace{0,\underbrace{1,2,3,\ldots}_{\mathbb{N}}}_{\mathbb{N}_0}\}=\mathbb{Z}$

(ii) rational numbers: $\mathbb{Q}=\frac{p}{q}$, $p,\ q\in\mathbb{Z};\ \ q\notin 0$

(iii) real numbers: $\mathbb{R}$



$\Longrightarrow$ real numbers are "dense"
many equations have real numbers as solutions e.g.: as solutions e.g.:

$$x^2-2=0 \quad\Rightarrow\quad x_{1/2}=\pm\sqrt{2}\approx\pm1.41421\ldots \tag{2.1}$$

$$\sin x=0 \quad\Rightarrow\quad x=n\pi,\quad n\in\mathbb{Z} \tag{2.2}$$

$$e^x-10=0 \quad\Rightarrow\quad x=\ln 10\approx 2.302585\ldots \tag{2.3}$$

But:

$$\left.\begin{array}{ll} x^2+1=0 & \Rightarrow x=? \\ e^x+10=0 & \Rightarrow x=? \\ \sin x=2 & \Rightarrow x=? \end{array}\right\}\text{ no real numbers as solutions!}$$

$\Longrightarrow$ Definition of "new" numbers!

(iv) complex numbers: $\mathbb{C}$
$\Rightarrow$ almost all equations do have solutions in $\mathbb{C}$ (even if they have no real solution)
$\Rightarrow$ simplification of calculations [complex e-function]

One new number is needed:

$$x^2+1=0 \quad\Leftrightarrow\quad x^2=-1 \tag{2.4}$$

$$\Leftrightarrow\quad x=\pm\underbrace{\sqrt{-1}}_{i}=\pm i \tag{2.5}$$

Definition 1 *i is a number which square yields -1*
$i^2=-1$ $\quad\cdot$ *only this number is enough*
$\cdot$ *$i=\sqrt{-1}$ not quite correct!*

$\rightarrow$ all quadratic equations have now (always two!) simple solutions e.g.:

$$x^2+2x+10 \;=\; 0 \tag{2.6}$$

$$\rightarrow x_{1/2} \;=\; -1\pm\sqrt{1-10}=-1\pm\sqrt{-9}=-1\pm\sqrt{-1\cdot 9} \tag{2.7}$$

$$=\; -1\pm 3\sqrt{-1}=-1\pm 3i \tag{2.8}$$

Test:

$$(-1+3i)^2+2(-1+3i)+10=1-9-6i-2+6i+10=0$$

same for: $-1-3i$
$\Rightarrow$ expressions such as "$-1+3i$" are solutions of equations

$\rightarrow$"$-1+3i$" is a new kind of number, a "complex number"
$i =$"imaginary unit"

$$\underbrace{-1}_{\text{real part}} \quad + \quad \overbrace{3}^{\text{imaginary part}} i \qquad \Rightarrow \text{A complex number has a real and an imaginary part}$$

$$\left.\begin{array}{lcl} \mathrm{Re}(-1+3i) & = & -1 \\ \mathrm{Im}(-1+3i) & = & 3 \end{array}\right\} -1+3i$$

<u>In general:</u>

Definition 2 $z = a + bi \quad \in \mathbb{C},\ a, b \in \mathbb{R}$
e.g. $z_1 = \sqrt{2} + 5i, \quad z_2 = \frac{5}{3} + \sqrt{3}i$
$\Rightarrow$ *complex numbers are pairs of real numbers with* $i^2 = -1$

$$z = a + bi = (\underbrace{a}_{\text{real}}, \underbrace{b}_{\text{imaginary part}}) = \underbrace{\begin{pmatrix} a \\ b \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\text{2D Vector with two basic vectors}}$$

Definition 3 $\mathrm{Re}\{z\} = a, \quad \mathrm{Im}\{z\} = b$