Math for the Movement of Rigid Bodies

Illustration

Here is a bit of what it takes to deal with arbitrary movements or translation plus rotation of a rigid body. The pictures are straight form Wikipedia.
Note that it is hard to express all that without equations

Motion in space of a rigid body, and the inertia matrix [edit]

The scalar moments of inertia appear as elements in a matrix when a system of particles is assembled into a rigid body that moves in three-dimensional space. This inertia matrix appears in the calculation of the angular momentum, kinetic energy and resultant torque of the rigid system of particles.[3][4][5][6][25]

For analysis of a spinning top, see Precession § Classical (Newtonian), and Euler's equations (rigid body dynamics).

Let the system of n particles, $P_i, i = 1, \dots, n$ be located at the coordinates $\mathbf{r}_i$ with velocities $\mathbf{v}_i$ relative to a fixed reference frame. For a (possibly moving) reference point $\mathbf{R}$, the relative positions are

$$\Delta\mathbf{r}_i = \mathbf{r}_i - \mathbf{R}$$

and the (absolute) velocities are

$$\mathbf{v}_i = \boldsymbol{\omega} \times \Delta\mathbf{r}_i + \mathbf{V_R}$$

where $\boldsymbol{\omega}$ is the angular velocity of the system, and $\mathbf{V_R}$ is the velocity of $\mathbf{R}$.

Angular momentum [edit]

Note that the cross product can be equivalently written as matrix multiplication by combining the first operand and the operator into a, skew-symmetric, matrix, $[\mathbf{b}]$, constructed from the components of $\mathbf{b} = (b_x, b_y, b_z)$:

$$\mathbf{b} \times \mathbf{y} \equiv [\mathbf{b}]\,\mathbf{y}$$

$$[\mathbf{b}] \equiv \begin{bmatrix} 0 & -b_z & b_y \\ b_z & 0 & -b_x \\ -b_y & b_x & 0 \end{bmatrix}.$$

The inertia matrix is constructed by considering the angular momentum, with the reference point $\mathbf{R}$ of the body chosen to be the center of mass $\mathbf{C}$:[3][6]

$$\begin{aligned} \mathbf{L} &= \sum_{i=1}^{n} m_i\, \Delta\mathbf{r}_i \times \mathbf{v}_i \\ &= \sum_{i=1}^{n} m_i\, \Delta\mathbf{r}_i \times (\boldsymbol{\omega} \times \Delta\mathbf{r}_i + \mathbf{V_R}) \\ &= \left(-\sum_{i=1}^{n} m_i\, \Delta\mathbf{r}_i \times (\Delta\mathbf{r}_i \times \boldsymbol{\omega})\right) + \left(\sum_{i=1}^{n} m_i\, \Delta\mathbf{r}_i \times \mathbf{V_R}\right), \end{aligned}$$

where the terms containing $\mathbf{V_R}$ $(= \mathbf{C})$ sum to zero by the definition of center of mass.

Then, the skew-symmetric matrix $[\Delta\mathbf{r}_i]$ obtained from the relative position vector $\Delta\mathbf{r}_i = \mathbf{r}_i - \mathbf{C}$, can be used to define,

$$\mathbf{L} = \left(-\sum_{i=1}^{n} m_i [\Delta\mathbf{r}_i]^2\right) \boldsymbol{\omega} = \mathbf{I_C}\boldsymbol{\omega},$$

where $\mathbf{I_C}$ defined by

$$\mathbf{I_C} = -\sum_{i=1}^{n} m_i [\Delta\mathbf{r}_i]^2,$$

is the symmetric inertia matrix of the rigid system of particles measured relative to the center of mass $\mathbf{C}$.

Kinetic energy [edit]

The kinetic energy of a rigid system of particles can be formulated in terms of the center of mass and a matrix of mass moments of inertia of the system. Let the system of n particles $P_i, i = 1, \ldots, n$ be located at the coordinates $\mathbf{r}_i$ with velocities $\mathbf{v}_i$, then the kinetic energy is[3][6]

$$E_K = \frac{1}{2}\sum_{i=1}^{n} m_i \mathbf{v}_i \cdot \mathbf{v}_i = \frac{1}{2}\sum_{i=1}^{n} m_i \left(\boldsymbol{\omega} \times \Delta\mathbf{r}_i + \mathbf{V}_C\right) \cdot \left(\boldsymbol{\omega} \times \Delta\mathbf{r}_i + \mathbf{V}_C\right),$$

where $\Delta\mathbf{r}_i = \mathbf{r}_i - \mathbf{C}$ is the position vector of a particle relative to the center of mass.

This equation expands to yield three terms

$$E_K = \frac{1}{2}\left(\sum_{i=1}^{n} m_i \left(\boldsymbol{\omega} \times \Delta\mathbf{r}_i\right) \cdot \left(\boldsymbol{\omega} \times \Delta\mathbf{r}_i\right)\right) + \left(\sum_{i=1}^{n} m_i \mathbf{V}_C \cdot \left(\boldsymbol{\omega} \times \Delta\mathbf{r}_i\right)\right) + \frac{1}{2}\left(\sum_{i=1}^{n} m_i \mathbf{V}_C \cdot \mathbf{V}_C\right)$$

The second term in this equation is zero because $\mathbf{C}$ is the center of mass. Introduce the skew-symmetric matrix $[\Delta\mathbf{r}_i]$ so the kinetic energy becomes

$$\begin{aligned} E_K &= \frac{1}{2}\left(\sum_{i=1}^{n} m_i \left([\Delta\mathbf{r}_i]\,\boldsymbol{\omega}\right) \cdot \left([\Delta\mathbf{r}_i]\,\boldsymbol{\omega}\right)\right) + \frac{1}{2}\left(\sum_{i=1}^{n} m_i\right) \mathbf{V}_C \cdot \mathbf{V}_C \\ &= \frac{1}{2}\left(\sum_{i=1}^{n} m_i \left(\boldsymbol{\omega}^T [\Delta\mathbf{r}_i]^T [\Delta\mathbf{r}_i]\,\boldsymbol{\omega}\right)\right) + \frac{1}{2}\left(\sum_{i=1}^{n} m_i\right) \mathbf{V}_C \cdot \mathbf{V}_C \\ &= \frac{1}{2}\boldsymbol{\omega} \cdot \left(-\sum_{i=1}^{n} m_i [\Delta\mathbf{r}_i]^2\right)\boldsymbol{\omega} + \frac{1}{2}\left(\sum_{i=1}^{n} m_i\right) \mathbf{V}_C \cdot \mathbf{V}_C. \end{aligned}$$

Thus, the kinetic energy of the rigid system of particles is given by

$$E_K = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{I}_C \boldsymbol{\omega} + \frac{1}{2} M \mathbf{V}_C^2.$$

where $\mathbf{I}_C$ is the inertia matrix relative to the center of mass and M is the total mass.

Resultant torque [edit]

The inertia matrix appears in the application of Newton's second law to a rigid assembly of particles. The resultant torque on this system is,[3][6]

$$\boldsymbol{\tau} = \sum_{i=1}^{n} (\mathbf{r_i} - \mathbf{R}) \times m_i \mathbf{a}_i,$$

where $\mathbf{a}_i$ is the acceleration of the particle P_i. The kinematics of a rigid body yields the formula for the acceleration of the particle P_i in terms of the position $\mathbf{R}$ and acceleration $\mathbf{A}_R$ of the reference point, as well as the angular velocity vector $\boldsymbol{\omega}$ and angular acceleration vector $\boldsymbol{\alpha}$ of the rigid system as,

$$\mathbf{a}_i = \boldsymbol{\alpha} \times (\mathbf{r}_i - \mathbf{R}) + \boldsymbol{\omega} \times \boldsymbol{\omega} \times (\mathbf{r}_i - \mathbf{R}) + \mathbf{A}_R.$$

Use the center of mass $\mathbf{C}$ as the reference point, and introduce the skew-symmetric matrix $[\Delta\mathbf{r}_i] = [\mathbf{r}_i - \mathbf{C}]$ to represent the cross product $(\mathbf{r}_i - \mathbf{C})\times$, to obtain

$$\boldsymbol{\tau} = \left(-\sum_{i=1}^{n} m_i [\Delta\mathbf{r}_i]^2\right)\boldsymbol{\alpha} + \boldsymbol{\omega} \times \left(-\sum_{i=1}^{n} m_i [\Delta\mathbf{r}_i]^2\right)\boldsymbol{\omega}$$

The calculation uses the identity

$$\Delta\mathbf{r}_i \times (\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \Delta\mathbf{r}_i)) + \boldsymbol{\omega} \times ((\boldsymbol{\omega} \times \Delta\mathbf{r}_i) \times \Delta\mathbf{r}_i) = 0,$$

obtained from the Jacobi identity for the triple cross product as shown in the proof below:

Proof [show]

Thus, the resultant torque on the rigid system of particles is given by

$$\boldsymbol{\tau} = \mathbf{I}_C \boldsymbol{\alpha} + \boldsymbol{\omega} \times \mathbf{I}_C \boldsymbol{\omega},$$

where $\mathbf{I}_C$ is the inertia matrix relative to the center of mass.