Science of General Deformation

Some More Special Cases

The tensile test uses <u>uniaxial stress</u>; pushing or pulling in just one direction This is a very special situation; the link above gets you there.

In real life, all kinds of forces may act from all kinds of directions on specimen with all kinds of shapes. Before I go into this most *general* case, we will give a few more *special* cases a quick look.

The figure below shows some more rather special stress states; you can for sure cook up some more yourself.
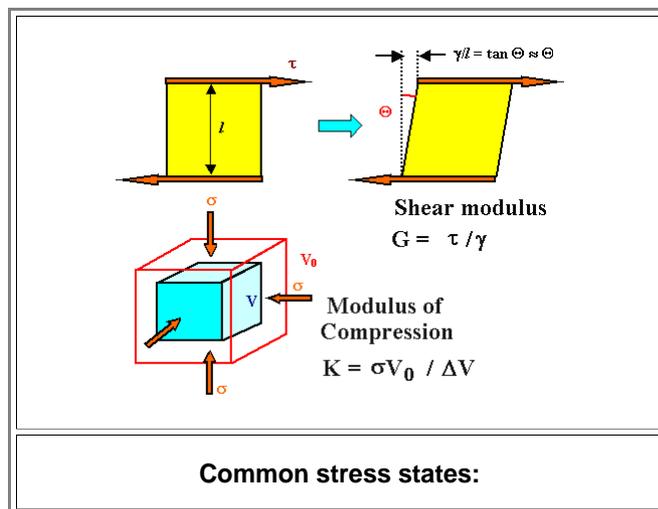

Common stress states:
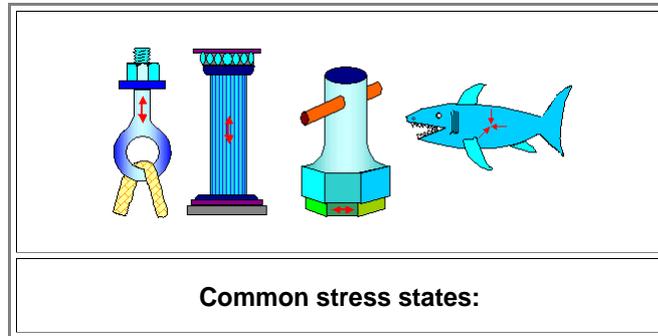
What we have from left to right is:

1. Uniaxial tensile stress
2. Uniaxial compressive stress
3. Pure shear,
4. Hydrostatic pressure

The first two I have already <u>dealt with</u>. The basic material parameter emerging from uniaxial testing was Young's modulus **Y** and Poisson's ratio ν.

Pure shear and uniform compression are shown below in a more scientific way. Pure shear means that the applied force is not at right angles to the surface it acts on, but lies in the surface. The shear stress, customarily abbreviated by τ and not σ, is still force divided by area.

Shear modulus
$$G = \tau / \gamma$$

Modulus of Compression
$$K = \sigma V_0 / \Delta V$$

Common stress states:

We can derive two new material parameters, a **shear modulus G** and a **compression modulus _K_** as shown.

Now we have four numbers describing the *elastic* deformation of some material Where will it end? I have good and bad news for you:

- *Good news:* Two numbers cover all **isotropic materials**.
- *Bad news:* For **anisotropic materials** you may need up to 21 numbers.

So what is an isotropic material as opposed to an anisotropic one? Simple: The properties of an isotropic material are independent of the direction. In other words, if you have some clump of material, let's say a potato, and now cut tensile test specimen out of it (something like french fries), you get the same test results no matter how you cut the specimen. Potatoes are isotropic materials; the taste of a french fry does not depend on how you cut one out of the volume.

- It is not the same for meat. Cutting with or across the grain makes a difference in mechanical properties and texture / taste.
  The same principle applies to crystals. In principle it matters how you cut your specimen. Crystals tend to be anisotropic. You know this. Graphite, for example, gives extremely easily in one direction and is extremely tough in another. That's why you can use it to lubricate things or to make extremely strong composites.

To describe properly what happens in a complex crystal when you subject it to mechanical stress, you need indeed 21 numbers then called "**elastic coefficients**". Sorry. But I have good news once more:

- All *cubic* crystals are isotropic - and that includes iron and steel.
- *Poly* crystals with a random distribution of grains are isotropic: Anisotropies simply average out.

- If that is true (it is) logic dictates that if I have *two* **elastic parameters**, I must be able to **calculate all the other ones**. That's what we do, here is one result, relating the four moduli discussed so far.

$$G = \frac{Y}{2(1 + \nu)} \approx 0{,}4\,E \quad \text{(for } \nu \approx 0{,}3)$$

$$K = \frac{Y}{3(1 - 2\nu)} \approx 0{,}8\,E \quad \text{(for } \nu \approx 0{,}3)$$

- The shear modulus, by the way, is the convenient modulus when we go for *plastic* instead of elastic deformation.

General Case of Deformation; Definition of Normal and Shear Stress

- In the most general case we have an arbitrarily shaped object that is not even uniform but possibly composed of different materials - e.g. a potato with rotten spots, your car, you.
  On this material act all kinds of forces from all kinds of directions. Forces are *vectors*, illustrated by arrows, and our potato about to be deformed might be illustrated like this:

One example of general stress acting on some inhomogeneous object
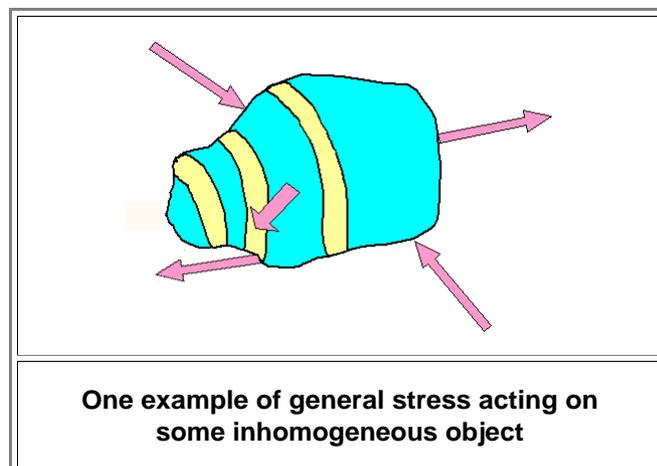
- We only want to *deform* the object. We don't want to move it and we don't want it to start spinning around some axis. That means that the vector sum of all forces must be zero and so must be the torsional moments.
  That makes the job somewhat easier but not really easy.

The first thing to realize is that the forces don't need to act exclusively perpendicular on some surface anymore (like in tensile testing) but may have any arbitrary angle. That means we must expand the definition of stress.

- For that we look at some force **F** acting at some angle on some surface **A**. If you like, you can describe the surface by a vector **A** that is perpendicular to the surface it describes and has a magnitude (= length) that gives the surface areas. The figure below gives an idea of that for some arbitrary internal area (in blue).

Normal and shear stresses on planes inclined to a force.
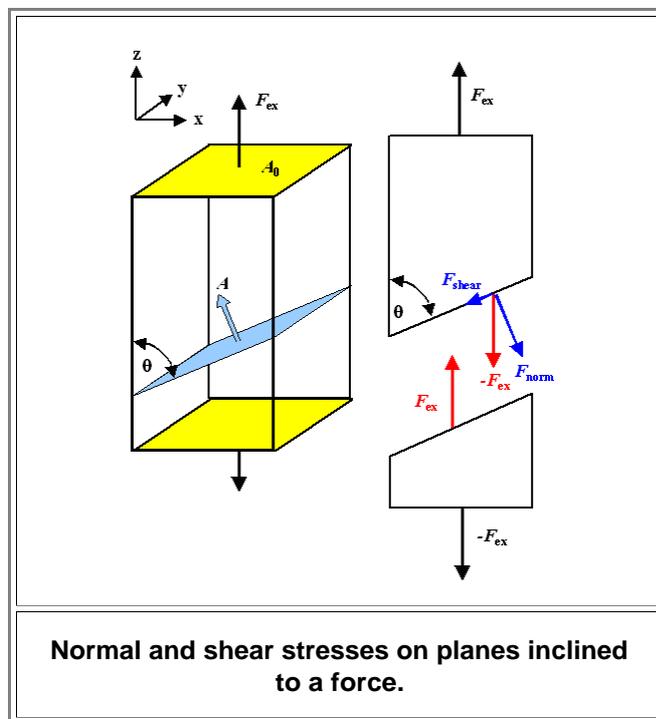
● We look at some plane inside an object inclined by some angle Θ to an acting force. We use this plane for a *virtual* cut through the object. Since the cut is supposed to leave everything exactly as before, we must put some forces on both sides that are chosen in such a way that exactly nothing happens. Since the external forces would move the two parts apart, these force must oppose the external forces F_{ex}. That implies that on our blue reference plane we must decompose F_{ex} into a component F_{normal} normal to the plane, and a component a F_{shear} contained in the plane.
That's rather easy to express in equations (provided you know the basics about vector math).

● Next we calculate the stresses acting on the chosen plane. We have a stress given by F_{normal} divided by the area of the plane, which we call normal stress and always abbreviate with σ, and a stress given by F_{shear} divided by the area of the plane, which we call shear stress and always abbreviate with τ.
The area of the plane contained inside the object is also easy to calculate.
For the forces and the area A we have

$$F_{norm} = F_{ex} \cdot \sin\Theta$$

$$F_{shear} = F_{ex} \cdot \cos\Theta$$

$$A = A_0/\sin\Theta$$

● With σ_{ex} = nominal stress applied to the object at large, the two stresses come out to

$$\sigma_{norm} = \frac{F_{norm}}{A} = \frac{F_{ex} \cdot \sin^2\Theta}{A_0} = \sigma_{ex} \cdot \sin^2\Theta$$

$$\sigma_{shear} = \frac{F_{shear}}{A} = \frac{F_{ex} \cdot \sin\Theta \cdot \cos\Theta}{A_0} = \frac{\sigma_{ex}}{2} \cdot \sin 2\Theta$$

▌ This was just doing some simple geometry. The results, however, are remarkable. To see this, we plot the normal and shear components of the forces and the stresses they cause vs. the angle Θ.

Normal and shear stresses on planes inclined to the force.
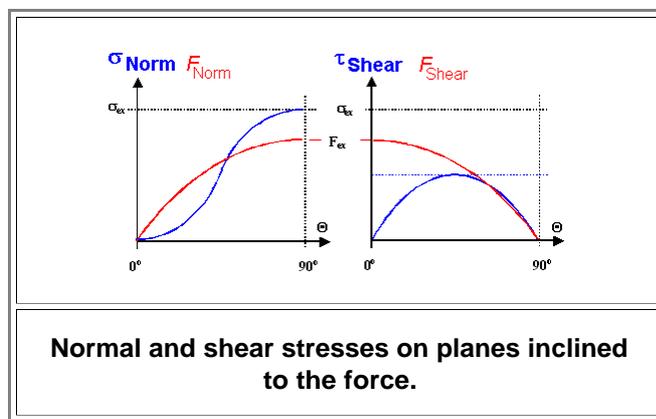
○ While the forces and the normal stress just change more or less smoothly between zero and their maximum values, the shear stress has a **maximum** for $\Theta = 45^0$ and is zero for $\Theta = 0; 90^0$.
Since, to get a little ahead of myself, plastic deformation depends on shear stresses, we see already here why so much happens on planes roughly at 45^0 to the direction of external stress. Those planes experience the maximum shear stress.

The long an short of this exercise was to establish what we do for the general case: some force acts on some plane. We decompose that force into its three basic components and then produce *three* stresses:

- A normal stress σ_{norm} given by the *one* normal component of the force divided by the area.
- Two shear stresses τ_{shear} given by the *two* components of the force in the plane divided by the area

Where that we will lead us we will see right below.

General Case of Deformation; Tensor Formulation

If we look back at the general problem sketched above, the first thing we do is to subdivide the potato or whatever else is under investigations into lots of (differentially) small cubes. Some forces act on all six sides of all these cubes. We don't know these forces offhand but must calculate them from the external forces acting on the outside of the potato. It ain't easy, so let's first look at what will happen in a general way.

○ Here is a complete picture of what one finds for *one* of all the many cubes we are now considering. On other cubes the direction and length a of all the arrows might be different but the general situation is the same.

General stress situation for any stressed (differentially) small cube.
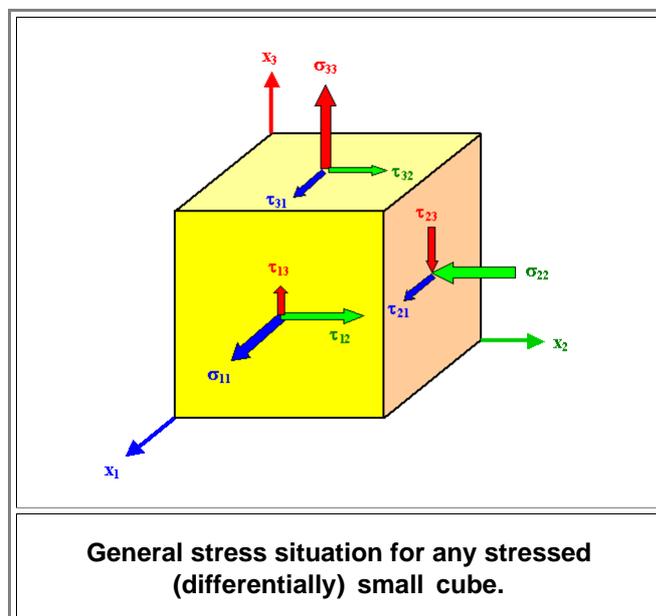
○ On each of the six planes we have three acting stresses; one normal stress and two shear stresses. Note that the arrows indicate the directions of the acting *force* components since stresses are *not* vectors - they are something worse - even so they are shown as arrows here.
A coordinate system with an x_1, x_2 and x_3-axes as shown allows to sort the whole mess; the rules are.

- $\sigma_{i,j}$ and $\tau_{i,j}$ denote normal and shear stresses, respectively.
- The indices run from **1** to **3** and have a defined meaning:
  i indicates the plane by giving the number of the coordinate axis perpendicular to it.
  j indicates the direction of the force component

- Here is an example: $\tau_{2,3}$ is the shear stress on the plane perpendicular to the x_2 axis caused by the force component in x_3 direction.

The thing to do now is to write all the components of the stress as follows:

$$\sigma_{ij}(x_1, x_2, x_3) = \begin{pmatrix} \sigma_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \sigma_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \sigma_{33} \end{pmatrix}$$

$$\sigma_{ij}(x_1, x_2, x_3) = \begin{pmatrix} \sigma_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \sigma_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \sigma_{33} \end{pmatrix}$$

- The nine numbers in this *matrix* describe all there is to know about the stress state in our potato or whatever object we consider at the point given by the coordinates (x_1, x_2, x_3).
- This reminds you (I hope) of a **vector**, a three-number thing that describes certain physical properties like velocities at some point in space.
  Good thinking. The nine-component matrix above is a kind of generalized vector called a **tensor** of second rank. Vectors are simply tensors of first rank, a **scalar** or simple number describing some physical properties like the temperature at some point is is a tensor of zero rank. Before I go a bit more into tensor math, we do some simplifications.

When we do deformation science, we usually assume that our object does not move or rotate. I squeeze my potato or whatever in such a way that its center of gravity stays put and it doesn't start to spin. That implies that the sum total of all forces and all momentums is zero. If we consider that in detail, the result is:

$$\sigma_{i\,j} \;=\; \sigma_{-i\,-j}$$

$$\tau_{i\,j} \;=\; \tau_{j\,i}$$

- There is some symmetry for the elements of the stress tensor. That is good - it means is that we only need to know *six* numbers an not nine to be able to write down the full **stress tensor** at some point. What a relief!

In many cases, like the ones illustrated above, the stress tensor has some symmetry that makes it simpler. Here are the examples from above plus a new one. One number is enough for four cases; only pure torsion or shear needs two.

$$\begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\sigma_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \tau_{12} & 0 \\ \tau_{12} & 0 & \tau_{12} \\ 0 & \tau_{12} & 0 \end{pmatrix} \begin{pmatrix} -\sigma_1 & 0 & 0 \\ 0 & -\sigma_1 & 0 \\ 0 & 0 & -\sigma_1 \end{pmatrix} \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Common stress states and the symmetry of the tensors going with them.

So what is a **tensor**? The most simple way to look at a tensor is to realize that it connects two vectors in the most general and simple way.

● Let's look at the definition of stress once more to see what that means. [We defined](#) stress originally as "force divided by the cross-sectional area" or $\sigma = F/A$ using a simple equation. The word "vector" never appeared and without even telling you it, I assumed that the force is always perpendicular to the plane it acts on.
But here we are a bit more sophisticated and realize that both F and A are vectors and that there is no need for them to be always parallel. Since you cannot divide a vector by a vector, the simple definition of stress in vector notation becomes.

$$\underline{F} = \sigma \cdot \underline{A}$$

● If σ would be a simple number or scalar, both vectors would have the same direction, just the length would be different. This is too simple, of course. In full generality direction and length could be different, meaning that every component of one vector will depend on all three components of the other vector. Written out we have:

$$F_x = \sigma_{xx} \cdot A_x + \sigma_{xy} \cdot A_y + \sigma_{xz} \cdot A_z$$
$$F_y = \sigma_{yx} \cdot A_x + \sigma_{yy} \cdot A_y + \sigma_{yz} \cdot A_z$$
$$F_z = \sigma_{zx} \cdot A_x + \sigma_{zy} \cdot A_z + \sigma_{zz} \cdot A_z$$

● We need a nine-number entity to connect two vectors in the *most simple* and general way. This thing we call a **tensor** (of second rank).

A hideous foreboding will now come up. "Where does this end", you might ask. "Never" is the answer you will get from your friendly mathematician. Materials Scientists will be more forthcoming and assure you that with tensors of fourth rank most everything can be done. Physicists tend to agree, but often invoke at least four-dimensions for "Einsteinian" space or even11 if they are given to string theory.

● You already know a tensor of fourth rank. In the backbone text I [defined](#) **Young's modulus** as "how much stress it takes to elongate a material by 0,01 %" or as stress divided by strain. Since in full generality Y relates two tensors of second rank, it must be a tensor of fourth rank! In equations (with the components of Y written as c_{ijhk} according to convention) that would be

$$\sigma = Y \cdot \epsilon \qquad \qquad \textbf{Simple equation}$$

$$\sigma_{11} = c_{11\,11} \cdot \epsilon_{11} + c_{11\,12} \cdot \epsilon_{12} + c_{11\,13} \cdot \epsilon_{13} + c_{11\,21} \cdot \epsilon_{21} + c_{11\,22} \cdot \epsilon_{22}$$
$$c_{11\,23} \cdot \sigma_{23} + c_{11\,31} \cdot \epsilon_{31} + c_{11\,32} \cdot \epsilon_{32} + c_{11\,33} \cdot \epsilon_{33}$$
$$\sigma_{12} = c_{12\,11} \cdot \epsilon_{11} + \dots.$$
$$\dots.. \qquad \qquad \textbf{Spelled out in components}$$

$$\sigma_{ij} = c_{ij\,kl} \cdot \epsilon_{kl} \qquad \textbf{Indices notation with "Einstein" convention:}$$
$$\textbf{always sum over common indices}$$

● When I showed that to my daughter (the one that studies literature), she remarked that she had never seen anything more hideous in all her life. Well, beauty lies in the eye of the beholder. Nevertheless, I assume that it will make you happy to know that the 81 components of Young's modulus proper (and then called **elastic coefficients**) boil down to

- 21 components at most for the case of [trigonal crystals](#).
- 2 components for the case of [cubic crystals](#).
- Two components for the case of (random) poly crystals.
- Between 2 and 21 for the rest

The long and short is that for iron and steel as for most other simple because cubic (and almost always poly-crystalline) metals, we get away with just two numbers. If we call them elastic coefficients, or Young's modulus plus Poission's ratio, or shear modulus or whatever, doesn't really matter. You always can convert one set into another one - if you know how to do it.

But don't kid yourself. What we just learned about Young's modulus applies to a lot of other material properties, typically given to innocents as a simple number. Take the index of refraction, for example, the dielectric "constant", or many others. They are actually tensors of various ranks and the do not always break down into just one or two numbers. Worse, the components of these tensors may be *complex* numbers, i.e. numbers that contain the square root of -1.
If you ever watched a three-dimensional movies, you had to wear some special glasses. When you use those glasses, you are looking straight through rather complex tensors, to put it this way. The tensor properties of materials are getting increasingly important and Material Scientists need to know about this.

OK - now that we got the the simple stuff out of the way, we can finally start with calculating how a potato deforms if you squeeze it.

Well, we won't. At least not here. Not only would I be hard pressed to do it, it is just too difficult. Answers to the simple everyday questions of how something deforms if subjected to forces are notoriously hard to get and the area of specialists with big computers and huge and complex software.
We will thus look only at some *very simple* stuff somewhat later.