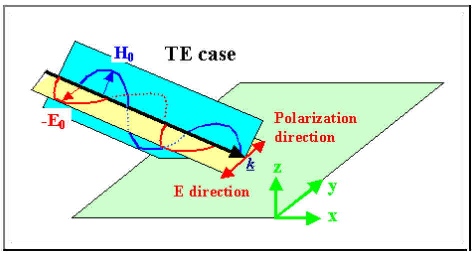
5.2.2 Fresnel Equations

Deriving the Fresnel Equations

Let's look at the TE mode (or _|_ mode) once more but now with the **coordinate system** needed for the equations coming up.



- The electrical field of the incoming beam thus writes as $\underline{E}_{in}$ **= (0,** E_{in}**, 0)**, i.e. there is only an oscillating component in ***y***-direction. For the ***y***-component E_{in} we can write $E_{in} = E_{in,0}\exp[-i(\underline{k}_{in} z\cos\alpha + \underline{k}_{in} x\sin\alpha)]$, decomposing the wave in an ***z*** and ***x*** component. We omitted the ωt phase factor because it will drop out anyway as soon as we go to intensities.
- Next we should write the corresponding equations for the reflected wave and the transmitted wave (requiring changes in the ***k***-vector).
- Then we need the same set of equations for the magnetic field. For that we have to know how the magnetic field of an electromagnetic wave can be derived from its electrical field. That means back to the Maxwell equations once more or for a taste of that to sub-chapter 5.1.4.
- After *you* did that you consider the *boundary conditions* as outlined before. Now you can start to derive the Fresnel equations. *You*, not me. It's tedious but good exercise. Let's just look at the general way to proceed.

First we write down the continuity of the tangential or here parallel component of $\underline{E}$ (and always same thing for $\underline{H}$ in principle). Since $\underline{E}$ has only components in ***y***-directions we have for those components

$$E_{in} + E_{ref} = E_{tr}$$

- While this looks a bit like the energy or *intensity* conservation equation from before, it is not! It is *completely different*, in fact! Our ***E***'s here are *field strengths* and *not* energy!

So let's look at the energy flux in ***z***-direction now, as given by the Poynting vector ***S***. It must be continuous since energy is neither genererated nor taken out at the interface as noted before. With the relation for energy from before and dropping the index "$_0$" for easier writing and reading, we obtain

$$\epsilon_1^{1/2}\,[(E_{in})^2 - (E_{ref})^2]\cdot\cos\alpha = \epsilon_2^{1/2}\,(E_{tr})^2\cdot\cos\beta$$

- This equation is simply the good old **Snellius law** in slight disguise (figure it out yourself, noting that $I_{tr} = I_{in} - I_{ref}$).

Dividing that equation by the one above it (remember: $(E_{in})^2 - (E_{ref})^2 = (E_{in} - E_{ref})\cdot(E_{in} + E_{ref})$ gives

$$\epsilon_1^{1/2}\,[E_{in} - E_{ref}]\cdot\cos\alpha = \epsilon_2^{1/2} E_{tr}\cdot\cos\beta$$

- With the good old relation $(\epsilon_1/\epsilon_2)^{1/2} = n_1/n_2 = \sin\alpha/\sin\beta$ (with α and β as given in the old figure for simplicity) and some shuffling of terms we finally obtain the **Fresnel equations for the TE case** .

Fresnel Equations TE case

$$E_{ref} = E_{in}\cdot\frac{\sin\beta\cos\alpha - \sin\alpha\cos\beta}{\sin\beta\cos\alpha + \sin\alpha\cos\beta} = -E_{in}\cdot\frac{\sin(\alpha-\beta)}{\sin(\alpha+\beta)}$$

$$E_{tr} = E_{in}\cdot\frac{2\sin\beta\cos\alpha}{\sin(\alpha+\beta)}$$

Going through the whole thing for the **TM** case (something I will not do here) gives the **Fresnel equations for the TM case**

Fresnel equations TM case

$$E_{ref} = E_{in} \cdot \frac{\sin\beta\cos\beta - \sin\alpha\cos\alpha}{\sin\beta\cos\beta + \sin\alpha\cos\alpha} = -E_{in} \cdot \frac{\tan(\alpha - \beta)}{\tan(\alpha + \beta)}$$

$$E_{tr} = E_{in} \cdot \frac{2\sin\beta\cos\alpha}{\sin(\alpha + \beta) \cdot \cos(\alpha - \beta)}$$

Relatively simple equations - but with a lot of power! Before we look at these equations a bit more closely, we modify them by dividing everything by E_{in}, so we get relative numbers (in % if you like) for the field strength or the relative intensities $(E/E_{in})^2$.

- The resulting numbers for the relative field strengths we call the **Fresnel coefficients**. We have four Fresnel coefficients: one each for reflection or transmittance, and that always separately for the **TE** and **TM** case.

Using the Fresnel Equations

A first extremely easy thing to do is to calculate the Fresnel coefficients for normal incidence ($\alpha = 0^o$). What we get for the standard case of going from air (less dense medium, $n = 1$ to some appreciable n (denser medium) for *both* the **TE** *and* **TM** case is

$$\frac{E_{ref}}{E_{in}} = -\frac{n - 1}{n + 1}$$

$$\frac{I_{ref}}{I_{in}} = \left(\frac{n - 1}{n + 1}\right)^2$$

- In other words, shining light straight on some glass with $n = 2$ means that almost **10 %** of the intensity will be reflected! This has immediate and dire consequences for optical instruments: you *must* provide some "anti-reflection" coating - otherwise your intensity gets too low after the light passed through a few lenses..

We need to do a bit of exercise here:

Exercise 5.2.1
Fresnel coefficients

- If we now speculate a little and consider metals as a material with *very large* dielectric constants and thus n, it is clear that they will reflect almost **100 %**.

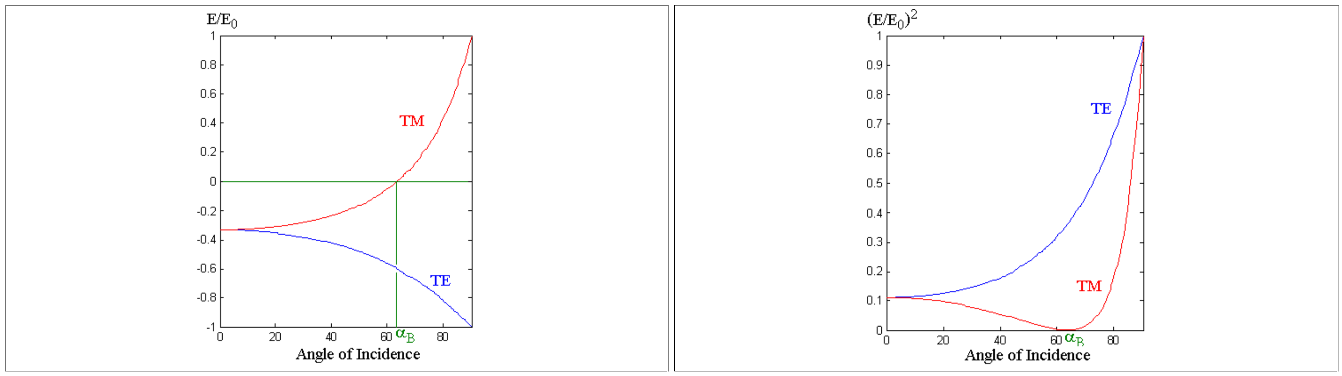
Next we plot the Fresnel coefficients as a function of α, the angle of incidence. We need four figures with **8** graphs to get the major points clear:

- Two figures showing the relative field strength (E_{ref}/E_{in}), always with two separate graphs for the two basic cases **TE** and **TM**.
  1. Case 1: $n_1 > n_2$, i.e. going from the less dense to the optically denser material
  2. Case 2: $n_1 < n_2$, i.e. going from the optically denser to the less dense material
- Two figures showing the relative *intensity* $(E_{ref}/E_{in})^2$, always with separate graphs for the two basic cases **TE** and **TM**; same cases as above

First we look at **case 1** with $n_1 < n_2$, i.e. going from the less dense to the optically denser material

- We take $n_1/n_2 = ½$ e.g. going from air with $n = 1$ into some glass with $n = 2$.
- Here are the **4** graphs for this case; we look at the reflected beam.

Case 1: Going from an optically less dense (n = 1) into a dense (n = 2) material



Electrical Field E/E_0 **Reflected beam** **Intensity** $(E/E_0)^2$

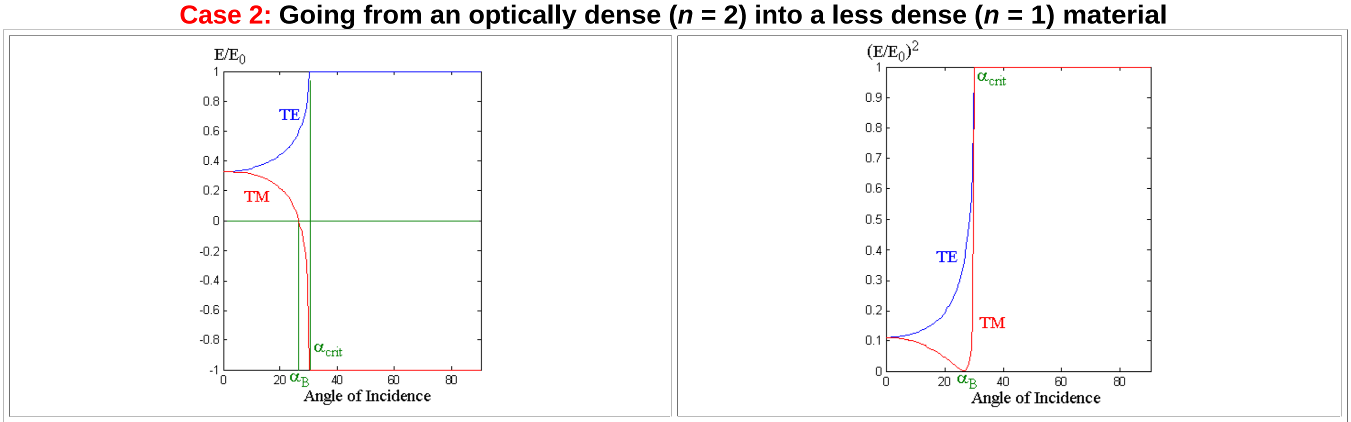
- Let's look at the field strength first What we see is

1. The numbers are negative for small α or almost perpendicular incidence. This means that we have a phase shift of **180°** between the incident and the reflected wave as outlined before.
2. The relative amplitude of the transmitted beam is simply **1 -** E_{ref}/E_{in}. It becomes small for large α's.
3. In the **TM** case the field strength is exactly zero at a certain angle α_B or "**Brewster angle**". This means that there is *no reflection* for this polarization; all the light will be transmitted.
4. So if the incident light consists of waves with arbitrary polarization, the component in **TM** direction will not be reflected and that means that whatever will be reflected must be polarized in TE direction. *We have a way to polarize light!*.
5. For **grazing incidence** or large α's almost all of the light will be reflected in either case.

- Looking a the intensities does not show anything new; you just see the "strength" of the reflected beam more clearly.

- Now let's look at **case 2** with $n_1 > n_2$, i.e. going from the more dense to the optically less dense material
  - We take $n_1/n_2 = 2$ e.g. going from some glass with $n = 2$ into air with $n = 1$.
  - Here are the **4** graphs for this case.

Case 2: Going from an optically dense (n = 2) into a less dense (n = 1) material



Electrical Field E/E_0 **Reflected Beam** **Intensity** $(E/E_0)^2$

- Let's look at the field strength first What we see is

1. The numbers are positive for small α or almost perpendicular incidence. This means that we have no phase shift of **180°** between the incident and the reflected wave as outlined before. Not that the reflected wave is the one staying inside the optically dense material.
2. The relative amplitude of the wave leaving the material is simply **1 -** E/E_0. It goes to zero rather quickly for increasing α's
3. In the **TM** case the field strength is exactly zero at a certain angle α_B or "**Brewster angle**". This means that there is *no reflection* for this polarization, all the light will be transmitted. Note that the value for the Brewster angle here is different from the one in the case going from the less dense to the more dense material.
4. So if the incident light consists of waves with arbitrary polarization, the component in **TM** direction will not be reflected and that means that whatever will be reflected must be polarized in **TE** direction. *We have a way to*

polarize light inside a material!

5. At some **critical angle** α_{crit} all light in either mode will be reflected. Beyond α_{crit} the Fresnel equations have only complex number solution and that means there is no field strength or energy outside the material. Light waves impinging at an angle **>**α_{crit} will be reflected right back into the material. α_{crit} is also known as the angel of **total reflection**.
6. That means that only light within a cone with opening angle **<** α_{crit} will be able to get of the material. It should be clear to you that a serious problem concerning **light emitting diodes,** (***LED***) is encountered here.

It's time for another exercise:

Exercise 5.2.2
Fresnel equations

Questionnaire
Multiple Choice questions to 5.2.2