4.10 Laminar flow through a round tube: Poiseuille's formula

Here we will discuss the laminar flow of a liquid through a straight tube (length l) showing a round cross section with radius R (cf. Fig. (4.7)). Liquid flow through the tube is enforced by a pressure difference $p_2 - p_1$. Friction within the liquid is described by the viscosity η. Due to the radial symmetry of the tube the flow pattern can only show a dependence on the position r (distance from the center of the tube). Exactly this is meant by "laminar", in contrast to "turbulent" where curls within the flow pattern exist.

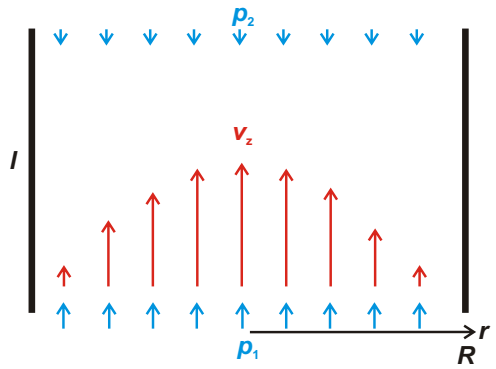


Figure 4.7: Momentum (speed) distribution in x direction between two walls of different speed in z direction.

The pressure force action on the outer surface of the liquid is

$$F_p = \pi r^2 (p_1 - p_2) \quad . \tag{4.74}$$

The friction force acting on an inner cylindrical surface $2\pi rl$ is

$$F_r = 2\pi rl\eta \frac{dv}{dr} \quad . \tag{4.75}$$

In steady state both forces compensate for each other, i.e. $F_r + F_p = 0$, leading to

$$\frac{dv}{dr} = \frac{\pi r^2 (p_2 - p_1)}{2\pi rl\eta} = \frac{r}{2\eta}\frac{p_2 - p_1}{l} \quad , \tag{4.76}$$

which after integration gives

$$v(r) = \frac{r^2}{4\eta}\frac{p_2 - p_1}{l} + C \quad . \tag{4.77}$$

Due to the sticking at the surface $v(R) = 0$ holds, so

$$C = -\frac{R^2}{4\eta}\frac{p_2 - p_1}{l} \quad , \tag{4.78}$$

finally giving

$$v(r) = \frac{R^2 - r^2}{4\eta}\frac{p_1 - p_2}{l} \quad . \tag{4.79}$$

The flow pattern thus is a paraboloid.
The liquid volume flowing per time t through the hollow cylinder between r and $r + dr$ is

$$dV = 2\pi r dr v(r) t \quad . \tag{4.80}$$

and thus through the whole tube

$$\begin{aligned} V &= \int_0^R 2\pi r dr v(r) t \\ &= 2\pi t \frac{p_1 - p_2}{4\eta l} \int_0^R r dr (R^2 - r^2) = \pi t \frac{p_1 - p_2}{8\eta l} R^4 \quad , \end{aligned} \tag{4.81}$$

leading to a liquid flow

$$\frac{V}{t} = \pi \frac{p_1 - p_2}{8\eta l} R^4 \quad , \tag{4.82}$$

which is Poiseuille's formula: applying the same pressure difference but halving the radius of the tube the flow reduces by a factor of 16. This tremendous effect of reducing the radius of a tube has to be taken into account, e.g. when designing small liquid flow systems or discussing micro-fluidity.