1.16 From the free energy to the inner energy

We will transform the Free Energy

$$F(V,N,T) = U(V,N,T) - S(V,N,T)T \quad , \tag{1.30}$$

with its total differential

$$dF = \mu dN - pdV - SdT \quad . \tag{1.31}$$

The Legendre transformation with respect to the temperature T leads to:

$$\frac{\partial F}{\partial T} = -S \quad , \tag{1.32}$$

$$F(V,N,T(S)) + ST(S) = U(V,N,T(S)) - ST(S) + ST(S) = U(V,N,S) \quad . \tag{1.33}$$

and

$$dU = \mu dN - pdV + TdS \quad . \tag{1.34}$$

Thus the inner energy itself is a thermodynamic potential which describes the isolated system completely. It depends only on extensive coordinates, since no external forces act on the system which would define an intensive parameter for the system.
Hint: Starting from $U(V,N,S)$ we can calculate $S(V,N,U)$. Within the framework of classical thermodynamics one can prove that S is a thermodynamic potential for this coordinates. A result which is even easier proved in statistical mechanics.