

Fourier Series and Transforms

Fourier Series

Basics

Every (physically sensible) **periodic** function $f(t) = f(t + T)$ with $T = 1/\nu = 2\pi/\omega$ and ν, ω = frequency and angular frequency, respectively, may be written as a **Fourier series** as follows

$$f(t) \approx a_0/2 + a_1 \cdot \cos \omega t + a_2 \cdot \cos 2 \omega t + \dots + a_n \cdot \cos n \omega t + \dots \\ + b_1 \cdot \sin \omega t + b_2 \cdot \sin 2 \omega t + \dots + b_n \cdot \sin n \omega t + \dots$$

and the Fourier coefficients a_k and b_k (with the index $k = 0, 1, 2, \dots$) are determined by

$$a_k = \frac{2}{T} \cdot \int_0^T f(t) \cdot \cos k \omega t \cdot dt \\ b_k = \frac{2}{T} \cdot \int_0^T f(t) \cdot \sin k \omega t \cdot dt$$

This can be written much more elegantly using [complex numbers](#) and functions as

$$f(t) = \sum_{-\infty}^{+\infty} c_n \cdot e^{in \omega t}$$

The coefficients c_n are obtained by

$$c_n = \int_0^T f(t) \cdot e^{-in \omega t} \cdot dt = \begin{cases} \frac{1}{2}(a_n - ib_n) & \text{for } n > 0 \\ \frac{1}{2}(a_{-n} + ib_{-n}) & \text{for } n < 0 \end{cases}$$

The function $f(t)$ is thus expressed as a sum of sin functions with the **harmonic frequencies** or simply **harmonics** $n \cdot \omega$ derived from the fundamental frequency $\omega_0 = 2\pi/T$.

The coefficients c_n define the **spectrum** of the periodic function by giving the amplitudes of the harmonics that the function contains.

Fourier Transforms

A nonperiodic function $f(t)$ ("well-behaved"; we are not looking at some **abominable** functions only mathematicians can think of) can also be written as a Fourier series, but now the Fourier coefficients have some values for all frequencies ω , not just for some harmonic frequencies.

Instead of a spectrum with defined lines at the harmonic frequencies, we now obtain a **spectral density** function $g(\omega)$, defined by the following equations

$$f(t) = \int_{-\infty}^{+\infty} g(\omega) \cdot e^{i\omega t} \cdot d\omega$$

$$g(\omega) = (1/2\pi) \cdot \int_{-\infty}^{+\infty} f(t) \cdot e^{-i\omega t} \cdot dt$$

🔷 The simplicity, symmetry and elegance (not to mention their usefulness) of these **Fourier integrals** is just amazing!