Einstein Coefficients

Advanced

- Again, we come back to the question: Do (direct) semiconductors glow in the dark?
  - The answer was yes – but only to the extent that all (black) bodies glow in the dark, following Plancks famous formula.
  - Here we will look at this question in a different way that also will allow us to obtain the Einstein coefficients.
- Instead of looking at the equilibrium distribution of all kinds of radiation in a "black body", we now consider only the frequencies prevalent in direct semiconductors, i.e. radiation with **h**$\nu \approx E_g$. We then have the three basic processes between electrons (and holes) and radiation:
- **Fundamental absorption**
  - The rate R_{fa} with which fundamental absorption takes place was given by (we use the simple version)

$$\boxed{R_{fa} = A_{fa} \cdot N_{eff}^2 \cdot u(\nu) \cdot \Delta\nu \cdot \left(1 - f_{h\,in\,V}(E^V, E_F^h, T)\right) \cdot \left(1 - f_{e\,in\,C}(E^C, E_F^e, T)\right)}$$

  - Since we now consider thermal equilibrium, we have $E_F{}^h = E_F{}^e = E_F$. We also can replace $1 - f_{h\,in\,V}(E^V, E_F^h, T)$ by $f(E, E_F, T)$ because the probability of *not* finding a hole at $E^V = E$ is equal to the probability of finding an electron; and $f_{h\,in\,V}(E^V, E_F^h, T)$ can be written as by $1 - f(E, E_F, T)$. Moreover, wherever we have $f_{e\,in\,C}$, we simply substitute by $f(E + h\nu, E_F, T)$. This yields

$$\boxed{R_{fa} = A_{fa} \cdot N_{eff}^2 \cdot u(\nu) \cdot \Delta\nu \cdot \left(f(E, E_F, T)\right) \cdot \left(1 - f(E + h\nu, E_F, T)\right)}$$

- **Stimulated emission**.
  - The rate R_{se} for stimulated emission (in the form rewritten for equilibrium exactly as above) was

$$\boxed{R_{se} = A_{se} \cdot N_{eff}{}^2 \cdot u(\nu) \cdot \Delta\nu \cdot \left(f(E + h\nu, E_F, T)\right) \cdot \left(1 - f(E, E_F, T)\right)}$$

- **Spontaneous emission**.
  - We have not yet considered the rate R_{sp} for spontaneous emission in the same formalism as the other two, but that is easy now. We have

$$\boxed{R_{sp} = B_{sp} \cdot N_{eff}^2 \cdot \left(f(E + h\nu, E_F, T)\right) \cdot \left(1 - f(E, E_F, T)\right)}$$

- Combining everything gives a surprisingly simple equation for R_{sp} :

$$\boxed{R_{sp} = \frac{R_{se} \cdot B_{sp}}{A_{se} \cdot u(\nu) \cdot \Delta\nu}}$$

- Thermodynamic equilibrium now demands that the number of photons produced must be equal to the number of photons absorbed. In other words, the sum of the emission rates must equal the absorption rate, or

$$\boxed{R_{se} + R_{sp} = R_{fa}}$$

  - Inserting the equation for R_{sp} yields

$$R_{fa} - R_{se} = \frac{R_{se} \cdot B_{sp}}{A_{se} \cdot u(\nu) \cdot \Delta\nu}$$

$$\frac{R_{fa}}{R_{se}} = \frac{B_{sp}}{A_{se} \cdot u(\nu) \cdot \Delta\nu}$$

- From this we obtain

$$u(\nu) \cdot \Delta\nu = \left(\frac{A_{se} \cdot R_{fa}}{B_{sp} \cdot R_{se}} - \frac{A_{se}}{B_{sp}} \right)^{-1}$$

- All we have to do now is to insert all the lengthy equations we derived for the rates. The math required for that is easy, but tedious.
  - For ease of writing we now drop all indices and functionalities which are not desparately needed, insert the equations for R_{fa} and R_{se}, and obtain

$$u(\nu) \cdot \Delta\nu = \frac{A_{se} \cdot N_{eff}^2 \cdot u \cdot \Delta\nu \cdot A_{fa} \cdot f(E) \cdot (1 - f(E + h\nu)}{A_{se} \cdot N_{eff}^2 \cdot u \cdot \Delta\nu \cdot B_{sp} \cdot f(E + h\nu) \cdot [1 - f(E)]} - \frac{A_{se}}{B_{sp}}$$

  - Now insert the Fermi distribution and shuffle once more - *good exercise!* - , and you get

$$u(\nu) \cdot \Delta\nu = \frac{B_{sp}}{A_{fa} \cdot \exp(h\nu/kT) - A_{se}}$$

- We now have an equation for the density of photons at some particular frequencies defined by the semiconductor. However, we have not made any specific assumptions about this frequency except that it is in thermodynamic equilibrium
  - This requires that $u(\nu) \cdot \Delta\nu$ obtained in this special way must be *precisely identical* to the radiation density as expressed in Plancks fundamental formula (which was derived in another advanced module) and we have

$$\frac{8\pi \cdot n_{ref}{}^3 (h\nu)^2}{h^3 \cdot c^3 \cdot \exp(h\nu/kT) - 1} \cdot d(h\nu) = \frac{B_{sp}}{A_{fa} \cdot \exp(h\nu/kT) - A_{se}}$$

- With this equation we have reached our goal and proved that

$$A_{fa} = A_{se}$$

  - *Can you see why?* Well - the equation thus must be valid at *all temperatures*. This is only possible if $A_{fa} = A_{se}$! Think about it!
- Using this equality we finally obtain

$$B_{sp} = \frac{8\pi \cdot n_{ref}^3 \cdot (h\nu)^2 \cdot A_{se}}{h^3 \cdot c^3}$$

- This is an important, if slightly sad equation. It says that the Einstein coefficient of spontaneous emission is some constant *times* the Einstein coefficient of stimulated emission *times* the *square of the frequency*.
  - In other words: At frequencies high enough, spontaneous emission always wins - *it will be hard to make an X-ray Laser!*
  - Unfortunately, the result we obtained does not change by doing more fancy math, e.g. by using the more precise equation for the transition rates from the advanced module. We have to live with it.
- We could go on now. After all, spontaneous emission is a recombination channel that we have treated before - in chapter 2 and chapter 5.
  - In any case we simply had for the net recombination rate $U = \Delta n/\tau$ and U was the net recombination rate. For the fraction that recombines via spontaneous radiation, we simply have to take the lifetime τ for that process and obtain
  - $U = \Delta n/\tau_{sp}$.
- On the other hand, the definition of the spontaneous emission rated from above can be rewritten as

$$\boxed{R_{sp} = B_{sp} \cdot n^e \cdot n^h}$$

  - because the effective density of states times the relevant Fermi distribution gives simply the density of electrons and holes in their bands.
  - The density of carriers we write, as ever so often, as

$$\boxed{\begin{aligned} n^e &= n^e{}_0 + \Delta n^e \\ n^h &= n^h{}_0 + \Delta n^h \\ n^e{}_0 \cdot n^h{}_0 &= n_i{}^2 \end{aligned}}$$

  - We then have the cases

$$\boxed{\Delta\, n^e = \Delta n << n^e{}_{0},\, n^h{}_0}$$

i.e almost equilibrium, and

$$\boxed{\Delta n >> n^e{}_0 ,\ n^h}$$

  - i.e. the high injection case.
- For the rate of spontaneous recombination, we then may distinguish the extreme cases of near equilibrium ($\Delta n \approx =$ **0**, and $\Delta n >> n_{min}$ and express this in rates of spontaneous recombination. For $\Delta n = 0$ we would have equilibrium with a recombination rate for the spontaneous recombination of

$$\boxed{R^{eq}{}_{sp} = B_{sp}\,(n^e{}_0 \cdot n^h{}_0)}$$

  - for $\Delta \approx 0$, or

$$\boxed{R^{eq}{}_{sp} = B_{sp} \cdot n_i{}^2}$$

- For non-equilibrium, which is the condition we are ususally considering, so we drop the index on R_{sp}, we have generally

$$\boxed{\begin{aligned} R_{sp} &= B_{sp}(n^e{}_0 + \Delta n) \cdot (n^h{}_0 + \Delta n) \\ &= B_{sp}[n_i{}^2 + \Delta n \cdot (n^e{}_0 + n^h{}_0 + \Delta n \\ &= R^{eq}{}_{sp} + B_{sp} \cdot \Delta n \cdot (n^e{}_0 + n^h{}_0 + \Delta n) \end{aligned}}$$

- R^{eq}_{sp} becomes negligible as soon as $\Delta n >> n_{min}$ which is *not* yet high injection and which we will have in *all* interesting cases. We thus finally approximately

$$\boxed{R_{sp} \approx B_{sp} \cdot \Delta n \cdot (n^e{}_0 + n^h{}_0 + \Delta n)}$$

Equating these expression with the simple formula $R_{sp} = \Delta n / \tau_{sp}$ under *all* conditions, we can now express the life time in terms of the Einstein coefficient and the carrier concentration.

- For low injection conditions, i.e. relatively small Δn meaning $R^{li}_{sp} \approx B_{sp} \cdot \Delta n \cdot (n^e{}_0 + n^h)$ we have

$$\boxed{\tau^{li}_{sp} = \frac{1}{B_{sp} \cdot (n^e{}_0 + n^h)}}$$

- For high injection, i.e $\Delta n >> n_{maj}$, meaning $R^{hi}_{sp} \approx B_{sp} \cdot \Delta n \cdot (\Delta n)$, we have

$$\boxed{\tau^{hi}_{sp} = \frac{1}{B_{sp} \cdot \Delta n}}$$

This compares favorably with our old Shockley-Read-Hall formula where we had

$$\boxed{\tau = \frac{1}{v \cdot \sigma \cdot n^{maj}}}$$

- with $\mathbf{v}$ = thermal velocity and σ = capture cross section .

Here some circle closes. But we will delve no more into this subject but simply remember: The Einstein coefficients of stimulated emission and fundamental absorption are identical for very fundamental reasons!