

2.11 Classification of $N \times N$ Matrices:

$$\tilde{A} = \begin{pmatrix} a_{11} & \cdots & a_{1N} \\ \vdots & & \vdots \\ a_{N1} & \cdots & a_{NN} \end{pmatrix}$$

- (i) $\det(\tilde{A}) \neq 0 \Rightarrow$ regular matrix
- (ii) $\tilde{A} = \tilde{A}^\top \Rightarrow$ symmetric matrix $a_{jk} = a_{kj}$ example $\begin{pmatrix} 1 & 0 & 4 \\ 0 & 2 & 2 \\ 4 & 2 & 3 \end{pmatrix}$
- (iii) $\tilde{A} = -\tilde{A}^\top \Rightarrow$ anti-symmetric matrix $a_{jk} = -a_{kj}$, in particular $a_{jj} = 0 \Rightarrow \text{tr}(\tilde{A}) = 0$
- (iv) $\tilde{A} = \tilde{A}^+ \Rightarrow$ self-adjointed matrix or Hermite-matrix, this means:

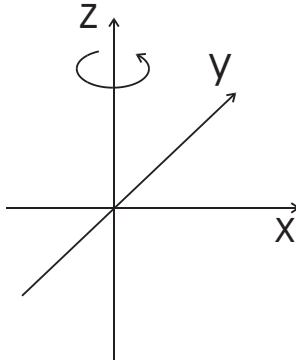
$$\tilde{A} = \overline{(\tilde{A}^\top)}, \quad a_{jk} = \overline{a_{kj}}$$

$$\text{example: } \tilde{A} = \begin{pmatrix} 1 & 2 & -i \\ 2 & 2 & 1-i \\ i & 1+i & 3 \end{pmatrix},$$

if $\tilde{A}$ is real then (ii) and (iv) are equivalent. $\rightarrow$ diagonal elements of a Hermite-matrix are real, because of $a_{jj} = \overline{a_{jj}}$. The determinant is also real, i.e. $\det(\tilde{A}) \in \mathbb{R}$ if $\tilde{A}$ is a Hermite matrix.

- (v) $\tilde{A} = -\tilde{A}^+ \Rightarrow$ anti-Hermite matrix ($\rightarrow$ diagonal elements vanish) in particular $a_{jj} = 0 \Rightarrow \text{tr}(\tilde{A}) = 0$
- (vi) $\tilde{A}^\top = \tilde{A}^{-1} \Rightarrow \tilde{A}$ is called orthogonal (real case) $\tilde{A}\tilde{A}^\top = \tilde{A}^\top\tilde{A} = \tilde{I}$ if $\tilde{A}$ is complex than $\tilde{A}^+ = \tilde{A}^{-1}$ means that $\tilde{A}$ is "unitary". Properties of orthogonal matrices: $\det(\tilde{A}) = \pm 1$ (follows from $\det(\tilde{A}) = \det(\tilde{A}^\top)$ and $\det(\tilde{A}\tilde{A}^\top) = \det(\tilde{A})\det(\tilde{A}^\top)$) if $\tilde{A}$ and $\tilde{B}$ orthogonal, then $\tilde{A} \cdot \tilde{B} = \tilde{C}$ is also orthogonal.

Example: rotation around z-axis



$$\begin{aligned} \tilde{A} &= \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \rightarrow \tilde{A}^\top &= \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \Rightarrow \tilde{A}\tilde{A}^\top &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{since } \cos^2 \phi + \sin^2 \phi = 1 \end{aligned}$$

Test:

$$\det(\tilde{A}) = 1 \begin{vmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{vmatrix} = +1 \quad \text{o.k.}$$

- (vii) diagonal Matrix:

$$\tilde{A} = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & \ddots & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & a_{NN} \end{pmatrix}$$

$$\det(\tilde{A}) = a_{11} \cdot a_{22} \cdot \dots \cdot a_{NN} \quad \text{for diagonal matrix } \tilde{A}$$