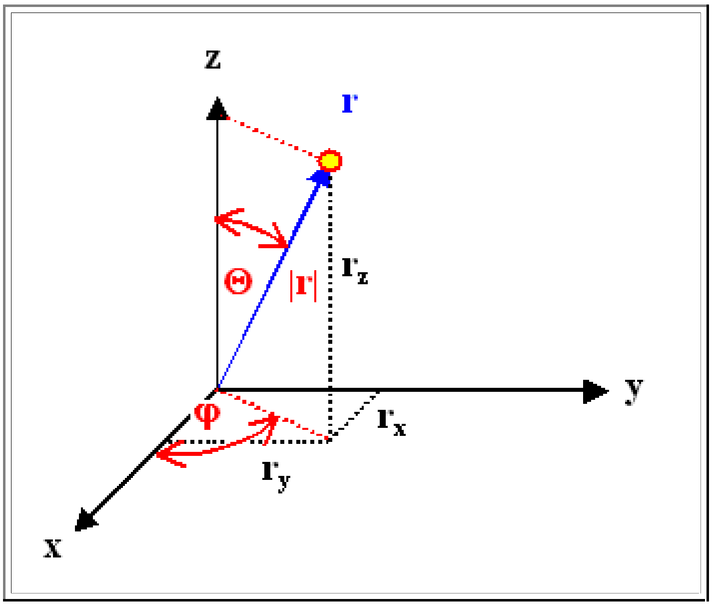
Spherical Coordinates

Basics

- For many mathematical problems, it is far easier to use spherical coordinates instead of Cartesian ones.
  - In essence, a vector ***r*** (*we drop the underlining here*) with the Cartesian coordinates **(*x,y,z*)** is expressed in spherical coordinates by giving its distance from the origin (assumed to be identical for both systems) |***r***|, and the two angles φ and Θ between the direction of ***r*** and the ***x***- and **z**-axis of the Cartesian system.
  - This sounds more complicated than it actually is: φ and Θ are nothing but the geographic ***longitude*** and ***latitude***. The picture below illustrates this.



- This is simple enough, for the translation from one system to the other one we have the equations

$$x = r \cdot \sin\Theta \cdot \cos\varphi \qquad r = (x^2 + y^2 + z^2)^{1/2}$$

$$y = r \cdot \sin\Theta \cdot \sin\varphi \qquad \varphi = \arctg\,(y/x)$$

$$z = r \cdot \cos\Theta \qquad \Theta = \arctg \frac{(x^2 + y^2 + z^2)^{1/2}}{z}$$

- Not particularly difficult, but not so easy either.
- Note that there is now a certain ambiguity: You describe the *same* vector for an ∞ set of values for Θ and ϕ, because you always can add **n·2 π (n = 1,2,3...)** to any of the two angles and obtain the same result.
  - This has a first consequence if you do an integration. Lets look at the ubiquitous case of normalizing a wave function ψ **(x,y,z)** by demanding that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi\,(x,y,z) \cdot dxdydz = 1$$

- In spherical coordinates, we have

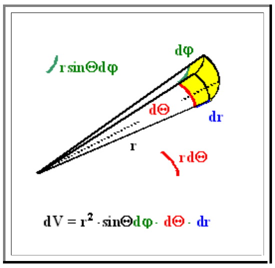
$$\int_{0}^{\infty}\int_{0}^{2\pi}\int_{0}^{\pi} \psi\,(r,\varphi,\Theta) \cdot dr\,d\varphi\,d\Theta = 1$$

  - You no longer integrate from **-**∞ to ∞ with respect to the angles, but from **0** to **2**π for φ and from **0** to π for Θ because this covers all of space. *Notice the different upper bounds!*

- Lets try this by computing the volume V_R of a sphere with radius **R**. This is always done by summing over all the differential volume elements **dV** inside the body defined by some equation
  - In Cartesian coordinates we have for the volume element **dV = dxdydz**, and for the integral:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \color{red}{???}\ \mathbf{dxdydz}$$

  - Well, if you can just *formulate* the integral, let alone solving it, you are already doing well!
- In spherical coordinates we first have to define the volume element. This is relatively easily done by looking at a drawing of it:



  - An incremental increase in the three coordinates by **dr**, **dφ**, and **dΘ** produces the volume element **dV** which is close enough to a rectangular body to render its volume as the product of the length of the three sides.
  - Looking at the basic geometry, the length of the three sides are identified as **dr**, **r · dΘ**, and **r · sinΘ · dφ**, which gives the volume element

$$dV \;=\; r^2 \cdot \sin\Theta \cdot dr \cdot d\,\Theta \cdot d\varphi$$

- The volume of our sphere thus results from the integral

$$V_r \;=\; \int_0^{\infty}\int_0^{2\pi}\int_0^{\pi} r^2 \cdot \sin\Theta \cdot dr\, d\,\varphi\, d\Theta \;=\; 2\pi \cdot \int_0^{\infty}\int_0^{\pi} r^2 \cdot \sin\Theta \cdot dr\, d\,\Theta \;=\; 2\pi \cdot [-\cos\pi + \cos 0] \cdot \int_0^{\infty} r^2 \cdot dr$$

$$V_r \;=\; 2\pi \cdot [2] \cdot 1/3R^3 \;=\; (4/3) \cdot \pi \cdot R^3 \qquad \color{red}{\textit{q.e.d.}}$$

  - Not extremely easy, but no problem either.
- Next, consider **differential operators**, like **div**, **rot**, or more general, ∇ and ∇^2 (= Δ).
  - Lets just look at Δ to see what happens. We have (for some function **U**)

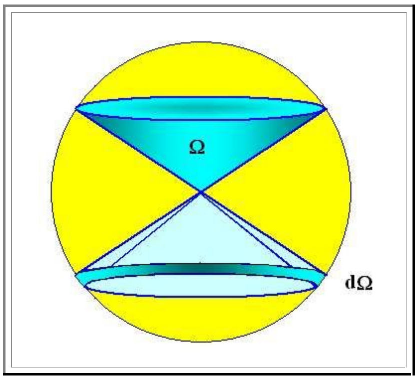
Cartesian coordinates
$\Delta = \dfrac{\partial^2 U}{\partial x^2} + \dfrac{\partial^2 U}{\partial y^2} + \dfrac{\partial^2 U}{\partial z^2}$

Spherical Coordinates
$\Delta = \dfrac{\partial^2 U}{\partial r^2} + \dfrac{2}{r} \cdot \dfrac{\partial U}{\partial\, r} + \dfrac{1}{r^2 \cdot \sin^2 \cdot \Theta} \cdot \dfrac{\partial^2 U}{\partial \varphi^2} + \dfrac{1}{r^2} \cdot \dfrac{\partial^2 U}{\partial\, \Theta^2} + \dfrac{\text{cotg}\,\Theta}{r^2} \cdot \dfrac{\partial U}{\partial \Theta}$

- Looks messy, **OK**, but it is still a lot easier to work with this Δ operator than with its Cartesian counterpart for problems with spherical symmetry; witness the solution of Schrödingers equation for the Hydrogen atom.

- Looking back now on our treatment of the orientation polarization, we find yet another way of expressing spherical coordinates for problems with particular symmetry:
  - We use a **solid angle** Ω and its increment **d**Ω.
  - A ***solid angle*** Ω is defined as the ratio of the area on a unit sphere that is cut out by a cone with the solid angle Ω to the total surface of a unit sphere ($= 4\pi R^2 = = 4\pi$ for $R = 1$).
  - A solid angle of 4π therefore is the same as the total sphere, and a solid angle of π is a cone with a (plane) opening angle of 120^o (figure that out our yourself).

- An incremental change of a solid angle creates a kind of ribbon around the opening of the cone defined by Ω. This is shown below



- Relations with spherical symmetry where the value of Θ does not matter - i.e. it does not appear in the relevant equations - are more elegantly expressed with the solid angle Ω.
  - That is the reason why practically all text books introduce Θ in the treatment of the polarization orientation. And in order to be compatible with most text books, that was what we did in the main part of the Hyperscript.
  - Of course, eventually, we have to replace Θ and **d** Θ by the basic variables that describe the problem, and that is only the angle δ in our problem (same thing as the angle φ here).

- Expressing **d**Θ in terms of δ is easy (compare the picture in the main text)
  - The radius ot the circle bounded by the **d**Θ ribbon is $r \cdot \sin\delta = \sin\delta$ because we have the unit sphere, and its width is simply **d**δ.
  - Its incremental area is thus the relation that we used in the main part.

$$\mathbf{d}\Theta = 2\pi \cdot \sin\delta \cdot \mathbf{d}\delta$$