

Stirlings Formula

Basics

- Stirlings formula is an indispensable tool for all combinatorial and statistical problems because it allows to deal with **factorials**, i.e. expressions based on the definition $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot \dots \cdot N := N!$
- It exists in several modifications; all of which are approximations with different degrees of precision. It is relatively easy to deduce its more simple version. We have

$$\ln x! = \ln 1 + \ln 2 + \ln 3 + \dots + \ln x = \sum_{y=1}^x \ln y$$

- With y = positive integer running from **1** to **x**
- For large y we may replace the sum by an integration in a good approximation and obtain

$$\sum_{y=1}^x \ln y \approx \int_1^x (\ln y) \cdot dy$$

- With $\int (\ln y) \cdot dy = y \cdot \ln y - y$, we obtain

$$\ln x! \approx x \cdot \ln x - x + 1$$

- This is the simple version of Stirlings formula. It can be even more simplified for large x because then $x + 1 \ll x \cdot \ln x$; and the most simple version, perfectly sufficient for many cases, results:

$$\ln x! \approx x \cdot \ln x$$

- However!!** We not only produced a simple approximation for $x!$, but turned a *discrete* function having values for integers only, into a *continuous* function, giving numbers for something like **3,141!** - which may or may not make sense.
- This may have dire consequences. Using the Stirling formula you may, e.g., move from *absolute probabilities* (always a number between **0** and **1**) to *probability densities* (any positive number) without being aware of it.
- Finally, an even better approximation exists (the prove of which would take some **20** pages) and which is already rather good for small values of x , say $x > 10$:

$$x! \approx (2\pi)^{1/2} \cdot x^{(x + 1/2)} \cdot e^{-x}$$