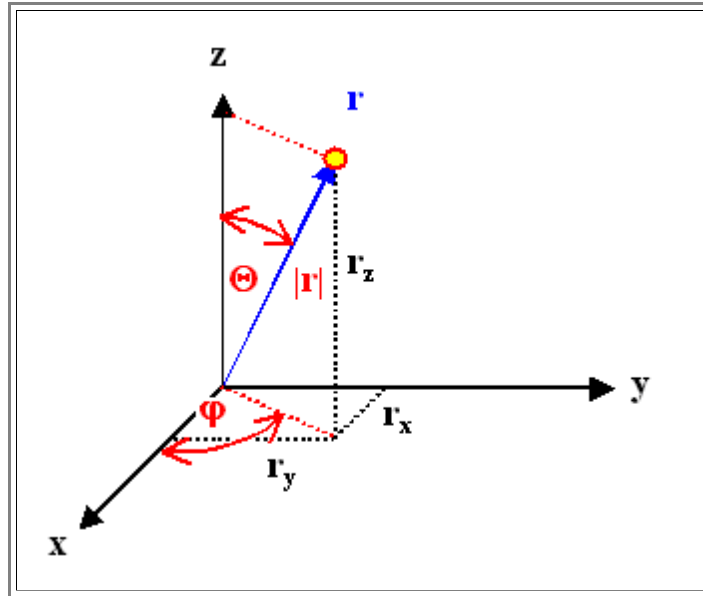


Spherical Coordinates

Basics

For many mathematical problems, it is far easier to use spherical coordinates instead of Cartesian ones.

- In essence, a vector \mathbf{r} (*we drop the underlining here*) with the Cartesian coordinates (x, y, z) is expressed in spherical coordinates by giving its distance from the origin (assumed to be identical for both systems) $|\mathbf{r}|$, and the two angles φ and Θ between the direction of \mathbf{r} and the x - and z -axis of the Cartesian system.
- This sounds more complicated than it actually is: φ and Θ are nothing but the geographic **longitude** and **latitude**. The picture below illustrates this.



This is simple enough, for the translation from one system to the other one we have the equations

$$\begin{aligned}
 x &= r \cdot \sin\Theta \cdot \cos\varphi & r &= (x^2 + y^2 + z^2)^{1/2} \\
 y &= r \cdot \sin\Theta \cdot \sin\varphi & \varphi &= \arctg(y/x) \\
 z &= r \cdot \cos\Theta & \Theta &= \arctg \frac{(x^2 + y^2 + z^2)^{1/2}}{z}
 \end{aligned}$$

Not particularly difficult, but not so easy either.

Note that there is now a certain ambiguity: You describe the **same** vector for an ∞ set of values for Θ and φ , because you always can add $n \cdot 2\pi$ ($n=1,2,3,\dots$) to any of the two angles and obtain the same result.

- This has a first consequence if you do an integration. Lets look at the ubiquitous case of normalizing a wave function $\psi(x, y, z)$ by demanding that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi(x, y, z) \cdot dx dy dz = 1$$

In spherical coordinates, we have

$$\int_0^{\infty} \int_0^{2\pi} \int_0^{\pi} \psi(r, \varphi, \Theta) \cdot dr d\varphi d\Theta = 1$$

- You no longer integrate from $-\infty$ to ∞ with respect to the angles, but from 0 to 2π for φ and from 0 to π for Θ because this covers all of space. *Notice the different upper bounds!*

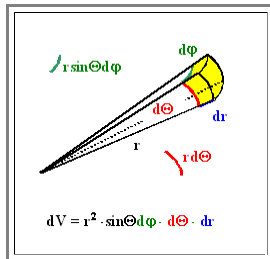
Let's try this by computing the volume V_R of a sphere with radius R . This is always done by summing over all the differential volume elements dV inside the body defined by some equation

- In Cartesian coordinates we have for the volume element $dV = dx dy dz$, and for the integral:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} ??? \, dx dy dz$$

- Well, if you can just *formulate* the integral, let alone solving it, you are already doing well!

In spherical coordinates we first have to define the volume element. This is relatively easily done by looking at a drawing of it:



- An incremental increase in the three coordinates by dr , $d\phi$, and $d\theta$ produces the volume element dV which is close enough to a rectangular body to render its volume as the product of the length of the three sides.
- Looking at the basic geometry, the length of the three sides are identified as dr , $r \cdot d\theta$, and $r \cdot \sin \theta \cdot d\phi$, which gives the volume element

$$dV = r^2 \cdot \sin \theta \cdot dr \cdot d\theta \cdot d\phi$$

The volume of our sphere thus results from the integral

$$V_R = \int_0^{\infty} \int_0^{2\pi} \int_0^{\pi} r^2 \cdot \sin \theta \cdot dr \, d\phi \, d\theta = 2\pi \cdot \int_0^{\pi} \int_0^{\infty} r^2 \cdot \sin \theta \cdot dr \, d\theta = 2\pi \cdot [-\cos \theta]_0^{\pi} \cdot \int_0^{\infty} r^2 \cdot dr$$

$$V_R = 2\pi \cdot [2] \cdot \frac{1}{3} R^3 = \frac{4}{3} \cdot \pi \cdot R^3 \quad \text{q.e.d.}$$

- Not extremely easy, but no problem either.

Next, consider **differential operators**, like **div**, **rot**, or more general, ∇ and $\nabla^2 (= \Delta)$.

- Let's just look at Δ to see what happens. We have (for some function U)

Cartesian coordinates

$$\Delta = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}$$

Spherical Coordinates

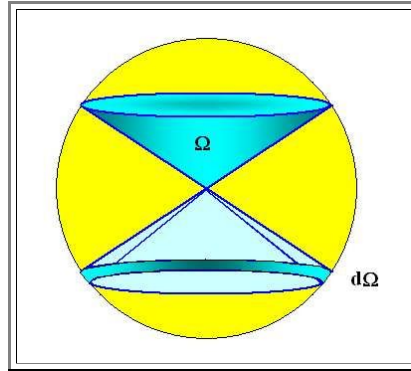
$$\Delta = \frac{\partial^2 U}{\partial r^2} + \frac{2}{r} \cdot \frac{\partial U}{\partial r} + \frac{1}{r^2 \cdot \sin^2 \theta} \cdot \frac{\partial^2 U}{\partial \phi^2} + \frac{1}{r^2} \cdot \frac{\partial^2 U}{\partial \theta^2} + \frac{\cotg \theta}{r^2} \cdot \frac{\partial U}{\partial \theta}$$

- Looks messy, **OK**, but it is still a lot easier to work with this Δ operator than with its Cartesian counterpart for problems with spherical symmetry; witness the [solution of Schrödingers equation for the Hydrogen atom](#).

Looking back now on our treatment of the orientation polarization, we find yet another way of expressing spherical coordinates for problems with particular symmetry:

- We use a **solid angle** Ω and its increment $d\Omega$.
- A **solid angle** Ω is defined as the ratio of the area on a unit sphere that is cut out by a cone with the solid angle Ω to the total surface of a unit sphere ($=4\pi R^2 == 4\pi$ for $R=1$).
- A solid angle of 4π therefore is the same as the total sphere, and a solid angle of π is a cone with a (plane) opening angle of 120° (figure that out our yourself).

An incremental change of a solid angle creates a kind of ribbon around the opening of the cone defined by Ω . This is shown below



Relations with spherical symmetry where the value of Θ does not matter - i.e. it does not appear in the relevant equations - are more elegantly expressed with the solid angle Ω .

- That is the reason why practically all text books introduce Θ in the treatment of the polarization orientation. And in order to be compatible with most text books, that was what we did in the main part of the Hyperscript.
- Of course, eventually, we have to replace Θ and $d\Theta$ by the basic variables that describe the problem, and that is only the angle δ in our problem (same thing as the angle φ here).

Expressing $d\Theta$ in terms of δ is easy (compare the [picture in the main text](#))

- The radius of the circle bounded by the $d\Theta$ ribbon is $r \cdot \sin\delta = \sin\delta$ because we have the unit sphere, and its width is simply $d\delta$.
- Its incremental area is thus the relation that we used in the [main part](#).

$$d\Theta = 2\pi \cdot \sin\delta \cdot d\delta$$